Problem 4.44

Show that Ψ' (Equation 4.197) satisfies the Schrödinger equation (Equation 4.191 with the potentials φ' and $\mathbf{A}'$ (Equation 4.196).

[TYPO: A parenthesis is missing after 4.191.]

Solution

Equation 4.191 on page 181 is the Schrödinger equation for a particle with mass m, charge q, and momentum $\mathbf{p}$ in the presence of electromagnetic fields, $\mathbf{E} = -\nabla\varphi - \partial\mathbf{A}/\partial t$ and $\mathbf{B} = \nabla \times \mathbf{A}$, where φ and $\mathbf{A}$ are given potential functions.

$$i\hbar\frac{\partial\Psi}{\partial t} = \left[\frac{1}{2m}(-i\hbar\nabla - q\mathbf{A})^2 + q\varphi\right]\Psi \tag{4.191}$$

Check to see if $\Psi' = e^{iq\Lambda/\hbar}\Psi$ satisfies this equation with $\varphi' = \varphi - \partial\Lambda/\partial t$ and $\mathbf{A}' = \mathbf{A} + \nabla\Lambda$, where $\Lambda = \Lambda(x, y, z, t)$. Note the vector product rule: $\nabla(fg) = f\nabla g + g\nabla f$.

$$i\hbar\frac{\partial\Psi'}{\partial t} \stackrel{?}{=} \left[\frac{1}{2m}(-i\hbar\nabla - q\mathbf{A}')^2 + q\varphi'\right]\Psi'$$

$$i\hbar\frac{\partial\Psi'}{\partial t} \stackrel{?}{=} \frac{1}{2m}(-i\hbar\nabla - q\mathbf{A}')^2\Psi' + q\varphi'\Psi'$$

$$i\hbar\frac{\partial}{\partial t}(e^{iq\Lambda/\hbar}\Psi) \stackrel{?}{=} \frac{1}{2m}[-i\hbar\nabla - q(\mathbf{A} + \nabla\Lambda)]^2(e^{iq\Lambda/\hbar}\Psi) + q\left(\varphi - \frac{\partial\Lambda}{\partial t}\right)(e^{iq\Lambda/\hbar}\Psi)$$

$$i\hbar\left(\frac{iq}{\hbar}\frac{\partial\Lambda}{\partial t}e^{iq\Lambda/\hbar}\Psi + e^{iq\Lambda/\hbar}\frac{\partial\Psi}{\partial t}\right) \stackrel{?}{=} \frac{1}{2m}[-i\hbar\nabla - q(\mathbf{A} + \nabla\Lambda)]^2(e^{iq\Lambda/\hbar}\Psi) + q\varphi e^{iq\Lambda/\hbar}\Psi - q\frac{\partial\Lambda}{\partial t}e^{iq\Lambda/\hbar}\Psi$$

$$\cancel{-q\frac{\partial\Lambda}{\partial t}e^{iq\Lambda/\hbar}}\Psi + i\hbar e^{iq\Lambda/\hbar}\frac{\partial\Psi}{\partial t} \stackrel{?}{=} \frac{1}{2m}[-i\hbar\nabla - q(\mathbf{A} + \nabla\Lambda)]^2(e^{iq\Lambda/\hbar}\Psi) + q\varphi e^{iq\Lambda/\hbar}\Psi - \cancel{q\frac{\partial\Lambda}{\partial t}e^{iq\Lambda/\hbar}}\Psi$$

$$i\hbar\frac{\partial\Psi}{\partial t} \stackrel{?}{=} \frac{e^{-iq\Lambda/\hbar}}{2m}[-i\hbar\nabla - q(\mathbf{A} + \nabla\Lambda)]^2(e^{iq\Lambda/\hbar}\Psi) + q\varphi\Psi$$

$$\frac{1}{2m}(-i\hbar\nabla - q\mathbf{A})^2\Psi + \cancel{q\varphi\Psi} \stackrel{?}{=} \frac{e^{-iq\Lambda/\hbar}}{2m}[-i\hbar\nabla - q(\mathbf{A} + \nabla\Lambda)]^2(e^{iq\Lambda/\hbar}\Psi) + \cancel{q\varphi\Psi}$$

$$e^{iq\Lambda/\hbar}(-i\hbar\nabla - q\mathbf{A})^2\Psi \stackrel{?}{=} [-i\hbar\nabla - q(\mathbf{A} + \nabla\Lambda)]^2(e^{iq\Lambda/\hbar}\Psi)$$

$$e^{iq\Lambda/\hbar}(i\hbar\nabla + q\mathbf{A})^2\Psi \stackrel{?}{=} [i\hbar\nabla + q(\mathbf{A} + \nabla\Lambda)]^2(e^{iq\Lambda/\hbar}\Psi)$$

$$\stackrel{?}{=} [i\hbar\nabla + q(\mathbf{A} + \nabla\Lambda)] \cdot [i\hbar\nabla(e^{iq\Lambda/\hbar}\Psi) + q(\mathbf{A} + \nabla\Lambda)(e^{iq\Lambda/\hbar}\Psi)]$$

$$\stackrel{?}{=} [i\hbar\nabla + q(\mathbf{A} + \nabla\Lambda)] \cdot \left[i\hbar\left(\Psi\nabla e^{iq\Lambda/\hbar} + e^{iq\Lambda/\hbar}\nabla\Psi\right)\right.$$
$$\left. + q(\mathbf{A} + \nabla\Lambda)(e^{iq\Lambda/\hbar}\Psi)\right]$$

Continue the simplification, noting the vector product rule: $\nabla\cdot(f\mathbf{C}) = f(\nabla\cdot\mathbf{C}) + \mathbf{C}\cdot\nabla f$.

$$
\begin{aligned}
e^{iq\Lambda/\hbar}(i\hbar\nabla + q\mathbf{A})^2\Psi &\stackrel{?}{=} [i\hbar\nabla + q(\mathbf{A}+\nabla\Lambda)]\cdot\left(i\hbar\Psi\nabla e^{iq\Lambda/\hbar} + i\hbar e^{iq\Lambda/\hbar}\nabla\Psi + q\mathbf{A}e^{iq\Lambda/\hbar}\Psi + qe^{iq\Lambda/\hbar}\Psi\nabla\Lambda\right) \\
&\stackrel{?}{=} [i\hbar\nabla + q(\mathbf{A}+\nabla\Lambda)]\cdot\left[i\hbar\Psi\left(\frac{iq}{\hbar}e^{iq\Lambda/\hbar}\nabla\Lambda\right) + i\hbar e^{iq\Lambda/\hbar}\nabla\Psi + q\mathbf{A}e^{iq\Lambda/\hbar}\Psi + qe^{iq\Lambda/\hbar}\Psi\nabla\Lambda\right] \\
&\stackrel{?}{=} [i\hbar\nabla + q(\mathbf{A}+\nabla\Lambda)]\cdot\left(\cancel{-qe^{iq\Lambda/\hbar}\Psi\nabla\Lambda} + i\hbar e^{iq\Lambda/\hbar}\nabla\Psi + q\mathbf{A}e^{iq\Lambda/\hbar}\Psi + \cancel{qe^{iq\Lambda/\hbar}\Psi\nabla\Lambda}\right) \\
&\stackrel{?}{=} [i\hbar\nabla + q(\mathbf{A}+\nabla\Lambda)]\cdot e^{iq\Lambda/\hbar}(i\hbar\nabla\Psi + q\mathbf{A}\Psi) \\
&\stackrel{?}{=} i\hbar\nabla\cdot[e^{iq\Lambda/\hbar}(i\hbar\nabla\Psi + q\mathbf{A}\Psi)] \\
&\qquad + q\mathbf{A}\cdot e^{iq\Lambda/\hbar}(i\hbar\nabla\Psi + q\mathbf{A}\Psi) + q\nabla\Lambda\cdot[e^{iq\Lambda/\hbar}(i\hbar\nabla\Psi + q\mathbf{A}\Psi)] \\
&\stackrel{?}{=} i\hbar\left\{e^{iq\Lambda/\hbar}\left[\nabla\cdot(i\hbar\nabla\Psi + q\mathbf{A}\Psi)\right] + (i\hbar\nabla\Psi + q\mathbf{A}\Psi)\cdot\nabla e^{iq\Lambda/\hbar}\right\} \\
&\qquad + iq\hbar e^{iq\Lambda/\hbar}(\mathbf{A}\cdot\nabla\Psi) + q^2e^{iq\Lambda/\hbar}(\mathbf{A}\cdot\mathbf{A})\Psi + iq\hbar e^{iq\Lambda/\hbar}(\nabla\Lambda\cdot\nabla\Psi) + q^2e^{iq\Lambda/\hbar}(\nabla\Lambda\cdot\mathbf{A})\Psi \\
&\stackrel{?}{=} i\hbar\left\{e^{iq\Lambda/\hbar}\left[i\hbar\nabla\cdot\nabla\Psi + q\nabla\cdot(\mathbf{A}\Psi)\right] + (i\hbar\nabla\Psi + q\mathbf{A}\Psi)\cdot\left(\frac{iq}{\hbar}e^{iq\Lambda/\hbar}\nabla\Lambda\right)\right\} \\
&\qquad + iq\hbar e^{iq\Lambda/\hbar}(\mathbf{A}\cdot\nabla\Psi) + q^2e^{iq\Lambda/\hbar}A^2\Psi + iq\hbar e^{iq\Lambda/\hbar}(\nabla\Lambda\cdot\nabla\Psi) + q^2e^{iq\Lambda/\hbar}(\nabla\Lambda\cdot\mathbf{A})\Psi \\
&\stackrel{?}{=} e^{iq\Lambda/\hbar}\left[-\hbar^2\nabla^2\Psi + iq\hbar\nabla\cdot(\mathbf{A}\Psi)\right] + (i\hbar\nabla\Psi + q\mathbf{A}\Psi)\cdot\left(-qe^{iq\Lambda/\hbar}\nabla\Lambda\right) \\
&\qquad + iq\hbar e^{iq\Lambda/\hbar}(\mathbf{A}\cdot\nabla\Psi) + q^2e^{iq\Lambda/\hbar}A^2\Psi + iq\hbar e^{iq\Lambda/\hbar}(\nabla\Lambda\cdot\nabla\Psi) + q^2e^{iq\Lambda/\hbar}(\nabla\Lambda\cdot\mathbf{A})\Psi \\
&\stackrel{?}{=} e^{iq\Lambda/\hbar}\left[-\hbar^2\nabla^2\Psi + iq\hbar\nabla\cdot(\mathbf{A}\Psi)\right] - \cancel{iq\hbar e^{iq\Lambda/\hbar}(\nabla\Psi\cdot\nabla\Lambda)} - \cancel{q^2e^{iq\Lambda/\hbar}(\mathbf{A}\cdot\nabla\Lambda)\Psi} \\
&\qquad + iq\hbar e^{iq\Lambda/\hbar}(\mathbf{A}\cdot\nabla\Psi) + q^2e^{iq\Lambda/\hbar}A^2\Psi + \cancel{iq\hbar e^{iq\Lambda/\hbar}(\nabla\Lambda\cdot\nabla\Psi)} + \cancel{q^2e^{iq\Lambda/\hbar}(\nabla\Lambda\cdot\mathbf{A})\Psi}
\end{aligned}
$$

Divide both sides by $e^{iq\Lambda/\hbar}$. Then expand the left side.

$$\begin{aligned}
(i\hbar\nabla + q\mathbf{A})^2\Psi &\overset{?}{=} -\hbar^2\nabla^2\Psi + iq\hbar\nabla\cdot(\mathbf{A}\Psi) \\
&\quad + iq\hbar(\mathbf{A}\cdot\nabla\Psi) + q^2A^2\Psi \\
(i\hbar\nabla + q\mathbf{A})\cdot(i\hbar\nabla\Psi + q\mathbf{A}\Psi) &\overset{?}{=} -\hbar^2\nabla^2\Psi + iq\hbar\nabla\cdot(\mathbf{A}\Psi) \\
&\quad + iq\hbar(\mathbf{A}\cdot\nabla\Psi) + q^2A^2\Psi \\
i\hbar\nabla\cdot(i\hbar\nabla\Psi + q\mathbf{A}\Psi) + q\mathbf{A}\cdot(i\hbar\nabla\Psi + q\mathbf{A}\Psi) &\overset{?}{=} -\hbar^2\nabla^2\Psi + iq\hbar\nabla\cdot(\mathbf{A}\Psi) \\
&\quad + iq\hbar(\mathbf{A}\cdot\nabla\Psi) + q^2A^2\Psi \\
-\hbar^2\nabla\cdot\nabla\Psi + iq\hbar\nabla\cdot(\mathbf{A}\Psi) + iq\hbar(\mathbf{A}\cdot\nabla\Psi) + q^2(\mathbf{A}\cdot\mathbf{A})\Psi &\overset{?}{=} -\hbar^2\nabla^2\Psi + iq\hbar\nabla\cdot(\mathbf{A}\Psi) \\
&\quad + iq\hbar(\mathbf{A}\cdot\nabla\Psi) + q^2A^2\Psi \\
-\hbar^2\nabla^2\Psi + iq\hbar\nabla\cdot(\mathbf{A}\Psi) + iq\hbar(\mathbf{A}\cdot\nabla\Psi) + q^2A^2\Psi &= -\hbar^2\nabla^2\Psi + iq\hbar\nabla\cdot(\mathbf{A}\Psi) \\
&\quad + iq\hbar(\mathbf{A}\cdot\nabla\Psi) + q^2A^2\Psi
\end{aligned}$$

Therefore, $\Psi' = e^{iq\Lambda/\hbar}\Psi$ satisfies Equation 4.191 with $\varphi' = \varphi - \partial\Lambda/\partial t$ and $\mathbf{A}' = \mathbf{A} + \nabla\Lambda$, where $\Lambda = \Lambda(x, y, z, t)$.